TEMPERATURE DISPLACEMENTS AND STRESSES IN GLASS-REINFORCED STRIP

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Glass-reinforced strip consists of a strand of straightened glass fibers arranged in a definite order and impregnated with resin. The physicomechanical properties of glass-reinforced strip determine the design properties of oriented glass-reinforced plastics. It is therefore of interest to study the stress distribution in the structure and the physicochemical properties of glass-reinforced strip on the basis of models of a structurally inhomogeneous body and the various properties of reinforcement and resin.

1. Let a volume element of the glass-reinforced strip be at a certain temperature $\theta = T - T_0$. We shall employ the very simple model illustrated in Fig. 1. In this model elastic rods are arranged at the nodes of a regular triangular net and run parallel with the x_1 axis to form a doubly periodic structure symmetrical with respect to the planes x_2 = const and x_3 = = const. The space between the fibers is filled with a viscoelastic resin.

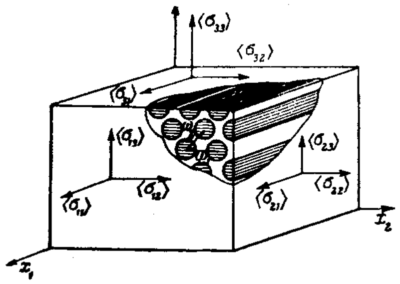

Fig. 1

Let $\omega_1 = 2$, $\omega_2 = \omega_1 e^{i\alpha}$ ($\alpha = 1/3\pi$) be the principal periods of the net, λ the dimensionless radius of the fibers, α_a, E_a, and ν_a are respectively the linear coefficient of thermal expansion, Young's modulus and Poisson's ratio for the fibers; α_s, E^*, and ν^* are the linear coefficient of thermal expansion, operator modules, and operator Poisson's ratio characterizing the viscoelastic properties of the resin,* u_i (i = 1, 2, 3) are the components of the displacement vector; henceforth quantities relating to the glass fibers will be denoted by the subscript "a" and quantities relating to the resin by the subscript "s."

The solution of the problem is constructed for a volume at a sufficient distance from the outer edge of the glass-reinforced strip; therefore in analyzing the deformation of the body in a constant temperature field it is necessary to take into account the fact that as a result of stress redistribution between reinforcement and resin with distance from the end faces of the strip cross sections x_1 = const remain plane on average

$$\langle \varepsilon_{11} \rangle = \alpha_s \theta + \langle \varepsilon_{11} \rangle_s = \alpha_a \theta + \langle \varepsilon_{11} \rangle_a, \quad \left(\langle \varepsilon_{11} \rangle = \frac{1}{F} \int_F dF \varepsilon_{11}, \quad F = \omega_1^2 \sin \alpha \right). \tag{1.1}$$

Here the symbol $\langle \varepsilon_{11} \rangle$ denotes the averaged value of ε_{11} over the area of the basic periodic parallelogram F, while $\langle \varepsilon_{11} \rangle_s$ and $\langle \varepsilon_{11} \rangle_a$ denote the average values of the still unknown strains of resin and reinforcement due to redistribution of the stresses at the end faces of the strip. Relations (1.1) hold for different tensile stresses along the x_1 axis for reinforcement and resin, while in the absence of an external force field the condition

$$\int_{F_a} dF \sigma_{11} + \int_{F_s} dF \sigma_{11} = 0 \tag{1.2}$$

must be satisfied.

The general solution of the problem of thermal expansion of the strip is composed of the solution of the problem of stretching along the x_1 axis, constructed without account for the interaction between resin and reinforcement, and the solution that does take this interaction into account with the body in the plane deformed state when $\langle \varepsilon_{11} \rangle = 0$.

The solution of the first problem is elementary:

$$\langle \sigma_{11} \rangle_a = E_a \langle \varepsilon_{11} \rangle_a, \qquad u_2 + iu_3 = -\nu_a z \langle \varepsilon_{11} \rangle + \alpha_a z \theta. \tag{1.3}$$

Analogous relations may be obtained for the resin

$$\langle \sigma_{11} \rangle_s = E^* \langle \varepsilon_{11} \rangle_s, \quad u_2 - iu_3 = -\nu^* z \langle \varepsilon_{11} \rangle_s + \alpha_s z \theta. \tag{1.4}$$

$$z = x_2 + ix_3.$$

The solution of the second problem is obtained with the aid [1] of the complex potentials $\Phi_\alpha(z, t)$, $\Psi_\alpha(z, t)$ and $\Phi_s(z, t)$, $\Psi_s(z, t)$. The boundary conditions at the contour of each fiber L_{mn} (for $z = \tau \equiv L_{mn}$; m, n = 0, ±1. . . .) for equal stresses and a given difference in the displacements of reinforcement and resin take the form [1]:

$$\Phi_s(\tau, t) + \overline{\Phi_s(\tau, t)} - e^{2i\vartheta} \{\bar{\tau} \Phi_s'(\tau, t) + \Psi_s(\tau, t)\} = \Phi_a(\tau, t) + \overline{\Phi_a(\tau, t)} - e^{2i\vartheta} \{\bar{\tau} \Phi_a'(\tau, t) + \Psi_a(\tau, t)\}, \tag{1.5}$$

$$(1 - G^*/G_a) \Phi_a(\tau, t) + (1 + G^*/G_a) \overline{\Phi_a(\tau, t)} - (1 - G^*/G_a) e^{2i\vartheta} \{\bar{\tau} \Phi_a'(\tau, t) + \Psi_a(\tau, t)\} - (\varkappa^* + 1) \overline{\Phi_s(\tau, t)} = 2G^* \{(\alpha_s - \alpha_a)(1 + \nu_a)\theta + (\nu_a - \nu^*) \langle \varepsilon_{11} \rangle_s\}. \tag{1.6}$$

For the structure in question it may be required to impose conditions (1.5) and (1.6) at only one arbitrarily chosen contour, if the potentials $\Phi_s(z, t)$ and $\Psi_s(z, t)$

*Linear forms of the relation between stresses and strains for resins are applicable at sufficiently low values of σ_{ik}. Thus for example, for an epoxy-maleic composition at T = 285°K the tensile stresses should not exceed $\sigma < 0.8\sigma_b$.

satisfy the relations of periodicity and symmetry

$$\Phi_s(z+\omega_j)=\Phi_s(z),$$
$$\Psi_s(z+\omega_j)=\Psi_s(z)-\bar{\omega}_j\Phi_s'(z)\qquad (j=1,\,2),\tag{1.7}$$

$$\Phi_s(\bar{z})=\overline{\Phi_s(z)},\quad \Phi_s(-z)=\Phi_s(z),$$
$$\Psi_s(\bar{z})=\overline{\Psi_s(z)},\quad \Psi_s(-z)=\Psi_s(z).\tag{1.8}$$

Conditions (1.7) and (1.8) are satisfied when the unknown functions are series of doubly periodic (elliptic) Weierstrass $\wp$ functions

$$\Phi_s(z,\,t)=2G^*\{(\alpha_s-\alpha_a)(1+\nu_a)\theta+(\nu_a-\nu^*)\langle\varepsilon_{11}\rangle_s\}\times$$
$$\times\Big[c_0(t)+\sum_{k=0}^{\infty}c_{2k+2}(t)\frac{\lambda^{2k+2}\wp^{(2k)}(z)}{(2k+1)}\Big],$$
$$\Psi_s(z,\,t)=2G^*\{(\alpha_s-\alpha_a)(1+\nu_a)\theta+(\nu_a-\nu^*)\langle\varepsilon_{11}\rangle_s\}\times\tag{1.9}$$
$$\times\Big[d_0(t)+\sum_{k=0}^{\infty}d_{2k+2}(t)\frac{\lambda^{2k+2}\wp^{(2k)}(z)}{(2k+1)}-\sum_{k=0}^{\infty}c_{2k+2}(t)\frac{\lambda^{2k+2}Q^{(2k+1)}(z)}{(2k+1)!}\Big].$$

Similarly, using the method of separation of variables, we get

$$\Phi_a(z,\,t)=2G^*\{(\alpha_s-\alpha_a)(1+\nu_a)\theta+$$
$$+(\nu_a-\nu^*)\langle\varepsilon_{11}\rangle_s\}\sum_{n=0}^{\infty}a_{2n}(t)\,z^{2n},\tag{1.10}$$
$$\Psi_a(z,\,t)=2G^*\{(\alpha_s-\alpha_a)(1+\nu_a)\theta+$$
$$+(\nu_a-\nu^*)\langle\varepsilon_{11}\rangle_s\}\sum_{n=0}^{\infty}b_{2n}(t)\,z^{2n}.$$

Using (1.5), (1.6) and the condition that at the edge of the basic parallelogram the principal vector of the forces is zero, we can determine the unknowns a_{2n}, b_{2n}, d_{2n}, and c_{2n} from the following system of equations:

$$a_0(t)=\frac{\eta+(\varkappa^*+1)S}{1+\eta+\xi\varkappa^*+\eta(\varkappa_a-1)G^*/G_a},$$
$$S=\sum_{k=1}^{\infty}c_{2k+2}(t)\lambda^{2k+2}\alpha_{0,\,k},$$
$$a_{2k}(t)=\frac{\varkappa^*+1}{1+\varkappa_aG^*/G_a}\sum_{n=0}^{\infty}c_{2n+2}(t)\lambda^{2n+2}\alpha_{k,\,n},\tag{1.11}$$
$$\lambda^{2k}b_{2k}(t)=-\frac{\varkappa^*+1}{1-G^*/G_a}c_{2k+2}(t)-(2k+1)\lambda^{2k+2}a_{2k+2}(t),$$
$$d_{2k+2}(t)=(2k+1)c_{2k}(t)-\frac{\varkappa^*-\varkappa_aG^*/G_a}{\varkappa^*+1}\lambda^{2k}a_{2k}(t),$$
$$c_0(t)={}^1/_2\xi d_2(t),\quad \xi c_2(t)=d_0(t),$$
$$c_0(t)=-\xi\frac{1+[\varkappa^*-1-(\varkappa_a-1)G^*/G_a]S}{1+\eta+\xi\varkappa^*+\eta(\varkappa_a-1)G^*/G_a},$$
$$\frac{\varkappa^*+G^*/G_a}{1-G^*/G_a}c_{2k}(t)=-\xi c_2(t)\delta_{k1}-\sum_{n=0}^{\infty}d_{2n+2}(t)\lambda^{2n+2k}\alpha_{k-1,\,n}+$$
$$+\sum_{n=0}^{\infty}c_{2n+2}(t)\,[(2n+2)\lambda^{2n+2k}\beta_{k-1,\,n}-(2k-1)\lambda^{2n+2+2k}\alpha_{k,\,n}]$$
$$(k=1,2,\ldots).$$

Here $\alpha_{i,k}$ and $\beta_{i,k}$ are the coefficients of the Laurent expansion [2] of the elliptic functions, $\xi=\pi\lambda^2/(\omega_1^2\sin\alpha)$ and $\eta=1-\xi$ are the volume ratios of reinforcement and resin.

From (1.1), (1.21, (1.9), (1.10) and (1.11) we get relations for the unknown strains

$$\langle\varepsilon_{11}\rangle_8=-(\alpha_s-\alpha_a)\,\xi\frac{E_a+8(1+\nu_a)(\nu_a-\nu^*)G^*a_0}{E^*}\theta,$$
$$\langle\varepsilon_{11}\rangle_a=(\alpha_s-\alpha_a)\Big\{1-\xi\frac{E_a+8(1+\nu_a)(\nu_a-\nu^*)G^*d_0}{E_1^*}\Big\}\theta,\tag{1.12}$$
$$E_1^*=\xi E_a+\eta E^*+8\xi(\nu_a-\nu^*)^2G^*a_0.$$

The displacements of the body in the plane x_1 = = const are found from the known equations [1]

$$u_2+iu_3=\alpha_sz\theta-\nu^*z\langle\varepsilon_{11}\rangle_s+(2G^*)^{-1}\times$$
$$\times\{\varkappa^*\varphi_s(z,\,t)-z\overline{\Phi_s(z,\,t)}-\overline{\psi_s(z,\,t)}\}\tag{1.13}$$
$$\Big(\varphi_s(z,\,t)=\int dz\,\Phi_s(z,\,t),\qquad \psi_s(z,\,t)=\int dz\,\Psi_s(z,\,t)\Big).$$

2. The general linear form of the relation between the stresses and strains for a body whose structure has three planes of symmetry may be written [3, 4] as

$$\langle\varepsilon_{11}\rangle=\frac{1}{E_1^*}\langle\sigma_{11}\rangle-\frac{\nu_{12}^*}{E_2^*}\langle\sigma_{22}\rangle-\frac{\nu_{13}^*}{E_3^*}\langle\sigma_{33}\rangle+\beta_1^*\theta,$$
$$\langle\varepsilon_{12}\rangle=\frac{1}{G_{12}^*}\langle\sigma_{12}\rangle,$$
$$\langle\varepsilon_{22}\rangle=-\frac{\nu_{21}^*}{E_1^*}\langle\sigma_{11}\rangle+\frac{1}{E_2^*}\langle\sigma_{22}\rangle-\frac{\nu_{23}^*}{E_3^*}\langle\sigma_{33}\rangle+\beta_2^*\theta,\tag{2.1}$$
$$\langle\varepsilon_{23}\rangle=\frac{1}{G_{23}^*}\langle\sigma_{23}\rangle,$$
$$\langle\varepsilon_{33}\rangle=-\frac{\nu_{31}^*}{E_1^*}\langle\sigma_{11}\rangle-\frac{\nu_{32}^*}{E_2^*}\langle\sigma_{22}\rangle+\frac{1}{E_3^*}\langle\sigma_{33}\rangle+\beta_3^*\theta,$$
$$\langle\varepsilon_{31}\rangle=\frac{1}{G_{31}^*}\langle\sigma_{31}\rangle.$$

Here E_i^*, ν_{sn}^*, G_{ik}^* are linear integral operators characterizing the viscoelastic properties of the inhomogeneous body, β_i^* are the operator coefficients of thermal expansion. The stresses and strains are averaged over areas containing a sufficiently large (over 1000) number of fibers; for the uniform stress state of a body with a regular doubly-periodic structure it is sufficient to carry out the averaging within the limits of the basic parallelogram.

The explicit form of the thermal expansion operator is found from (1.1), (1.2), and (2.1) for $\langle\sigma_{ik}\rangle=0$:

$$\beta_1^*=\alpha_s-(\alpha_s-\alpha_a)\,\xi\frac{E_a+8(1+\nu_a)(\nu_a-\nu^*)G^*a_0}{E_1^*}.\tag{2.2}$$

In order to determine β_2^*, β_3^* we must examine the average temperature displacements in the plane x_1 = = const

$$\langle u_2-iu_3\rangle={}^1/_2z(\beta_2^*+\beta_3^*)\theta+{}^1/_2\bar{z}(\beta_2^*-\beta_3^*)\theta.\tag{2.3}$$

If we compare the increments in the mean displacements according to (2.3) and (1.13) as we move from the point z to $z+\omega_j$, then

$${}^1/_2\omega_j(\beta_2^*+\beta_3^*)\theta+{}^1/_2\bar{\omega}_j(\beta_2^*-\beta_3^*)\theta=\omega_j\alpha_s\theta-\omega_j\nu^*\langle\varepsilon_{11}\rangle_s+$$
$$+\{(\alpha_s-\alpha_a)(1+\nu_a)\theta+(\nu_a-\nu^*)\langle\varepsilon_{11}\rangle_s\}[(\varkappa^*\omega_j-\omega_j+\tag{2.4}$$
$$+2\bar{\rho}_j)c_0-\xi(\varkappa^*\rho_j+\bar{\omega}_j)c_2],$$
$$(\rho_1=1,\ \rho_2=e^{-i\alpha}).$$

Setting j = 1, 2, we find the operator coefficients of thermal expansion of the inhomogeneous body

$$\beta_2{}^* = \alpha_s + (\alpha_s - \beta_1{}^*)\,\nu_{21}{}^* - (\alpha_s - \alpha_a)(1+\nu_a)(\nu^* - \nu_{21}{}^*)/(\nu^* - \nu_a),$$
$$\beta_3{}^* = \alpha_s + (\alpha_s - \beta_1{}^*)\,\nu_{31}{}^* - (\alpha_s - \alpha_a)(1+\nu_a)(\nu^* - \nu_{31}{}^*)/(\nu^* - \nu_a). \qquad (2.5)$$

The operator coefficients $\nu_{21}{}^*$ and $\nu_{31}{}^*$ are found from a study of the displacements when the strip is subjected to tensile stresses $\langle\sigma_{11}\rangle$ = const

$$\nu_{21}{}^* = \nu^* - (\nu_a - \nu^*)(\varkappa^* + 1)(c_0 - \xi c_2),$$
$$\nu_{31}{}^* = \nu^* - (\nu_a - \nu^*)(\varkappa^* + 1)(c_0 + \xi c_2) \quad (\text{при } \alpha = {}^1/_3\pi,\ c_2 = 0). \qquad (2.6)$$

The approximate value of $\nu_{21}{}^*$ (correct to 1%) is

$$\nu_{21}{}^* = \nu_{31}{}^* \approx \nu^* - \frac{\xi(\nu^* - \nu_a)(\varkappa^* + 1)}{1 + \eta + \xi\varkappa^* + \eta(\varkappa_a - 1)G^*/G_a}.$$

3. For a purely elastic resin and reinforcement all the operator quantities in the above equations are replaced by the elastic constants. Note that the physicomechanical properties of glasses in the temperature range to 500° K, and even higher for some special glasses, show only slight variation, whereas the properties of resins vary substantially at these temperatures. Therefore the given case is of interest for estimating the effect of the viscoelastic properties of the polymer on the coefficients of thermal expansion of a composite material. Figure 2 shows curves characterizing the change in the coefficients of thermal expansion with the volume content of glass reinforcement. For the type of structure in question the values of β_2 and β_3 are the same.

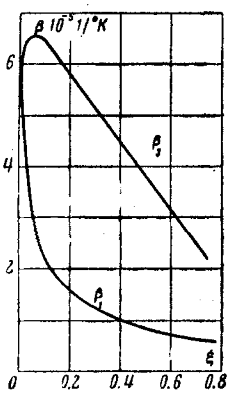

Fig. 2

Clearly, on average the value of β_1 is four times less than that of β_2 as a result of the constraints imposed by the glass fibers on displacements along the reinforcement. All the calculations were made for glass-reinforced strip made with aluminoborosilicate fibers (ν_a = = 0.2, E_a = 0.981 · 7 · 10^5 [bar], α_a = 0.49 · 10^{-5} [1/° K] and epoxy-maleic resin (ν_0 = 0.382, E_0 = 0.981 · 0.315 · 10^5 [bar], α_s = 6 · 10^{-5} [1/° K]).

The distribution of structural temperature stresses in the tape at the resin-glass boundary is shown in Fig. 3. Curves 1, 2, and 3 depict the dependence of the normal σ_n (at ξ = 0.736 and ξ = 0.227) and shear $\sigma_{n\vartheta}$ (at ξ = 0.736) stresses on the angle of orientation of the area element ϑ. The average stresses on areas perpendicular to the reinforcement are

$$\langle\sigma_{11}\rangle_a = E_a\langle\varepsilon_{11}\rangle + 8\nu_a G_0\{(\alpha_s - \alpha_a)(1+\nu_a)\theta + (\nu_a - \nu_0)\langle\varepsilon_{11}\rangle_s\}\,a_0$$

$$\langle\sigma_{11}\rangle_s = E_0\langle\varepsilon_{11}\rangle_s - 8\nu_0 G_0\{(\alpha_s - \alpha_a)(1+\nu_a)\theta + (\nu_a - \nu_0)\langle\varepsilon_{11}\rangle_s\}\,a_0\xi/\eta.$$

Below we present numerical values of the stresses for ξ = 0.736 $\langle\sigma_{11}\rangle_a$ = 0.981 · 0.59 bar, $\langle\sigma_{11}\rangle_s$ = − 0.981 · 2.63 θ bar, for ξ = 0.227 $\langle\sigma_{11}\rangle_a$ = 0.981 · 6.1 θ bar, $\langle\sigma_{11}\rangle_s$ = − 0.981 · 1.76 bar.

4. On the basis of simple creep tests on epoxy-maleic specimens at T = 285° K it was found that for sufficiently low stresses the viscoelastic properties of the resin can be described using the elastic memory theory. The simplest relations are obtained by using exponential-fractional functions as kernels. In this case we take [5]

$$E^* = E_0\{1 - \omega_0 \vartheta^*_{1-\lambda}(-\omega)\}, \qquad \nu^* = \nu_0\left\{1 + \frac{1 - 2\nu_0}{2\nu_0}\vartheta^*_{1-\lambda}(-\omega)\right\} \qquad (4.1)$$

where

$$\vartheta^*_{1-\lambda}(-\omega)\,f(t) = \int_0^t dt' f(t')(t - t')^{\lambda-1}\sum_{k=0}^{\infty}\frac{(-\omega)^k (t - t')^{k\lambda}}{\Gamma[(k+1)\lambda]}. \qquad (4.2)$$

In the given case we found

$$\omega = \omega_0 + \omega_\infty, \quad \lambda = 0.5, \quad \omega_0/\omega = 0.302, \quad \omega = 0.172\ \text{hour}^{-\lambda}.$$

At higher temperatures there is a substantial change in the parameters characterizing the viscoelastic properties of the resins and it is necessary to take into account the change in α_s.

A study of system (1.11) shows that for glass-reinforced plastics, when $G_s/G_a \ll 1$, the unknowns $c_{2k}(t)$ vary only slightly with time (except for $c_0(t)$, therefore, to a good approximation (of the order of 3%) we can set c_{2k} = const.

With an accuracy of better than 1% we can assume that the coefficient β_1 is constant. If we take into account the above-mentioned approximations, the explicit value of the operator $\beta_2{}^*$ will be

$$\beta_2{}^* = \beta_2{}^\circ + (\Omega_1 - \Omega_2)\,\nu_{21}{}^\circ \times$$
$$\times\left\{\alpha_s - \beta_1 + (\alpha_s - \alpha_a)\frac{(1+\nu_a)(2g - \Omega_2)}{(\nu_0 - \nu_a)(g - \Omega_2)}\right\}\vartheta^*_{1-\lambda}(-\omega_\infty - \Omega_2) +$$
$$+ (\alpha_s - \alpha\ )\frac{(1+\nu_a)\,g\omega_0}{g - \omega_0}\vartheta^*_{1-\lambda}(-\omega) + (\alpha_s - \alpha_a)\frac{1+\nu_a}{\nu_0 - \nu_a}\times$$
$$\times\, g\left\{\nu_0 - \nu_{21}{}^\circ - \frac{{}^1/_2 - \nu_0}{g - \omega_0}\omega_0 + \frac{1}{g - \Omega_2}\right\}\vartheta^*_{1-\lambda}(-\omega_\infty - g).$$

Here $\beta_2{}^\circ$ is the "instantaneous" value of β_2,

$$\Omega_1 = \frac{\omega_0}{2} + \frac{\xi\nu_a\omega_0}{2\xi\nu_a + \nu_0[1 - \xi(1 + 2\nu_a) + \eta(1 - 2\nu_a)G_0/G_a]},$$

$$g = \omega_0\frac{{}^1/_2 - \nu_0}{\nu_0 - \nu_a}, \qquad \Omega_2 = \omega_0\frac{1 + \eta(1 - 2\nu_a)G_0/G_a}{1 + \xi(1 - 2\nu_0) + \eta(1 - 2\nu_a)]G_0/G_a}.$$

To estimate the effect of the viscoelastic properties of the resin on $\beta_2{}^*$, we made a comparison of the value $\beta_2{}^*$ · 1 at the initial instant (for ξ = 0.736, $\beta_2{}^\circ$ = 2.18 · 10^{-5} [1/° K]).

As may be seen from the data, disregarding this effect leads to an error of the order of 10%.

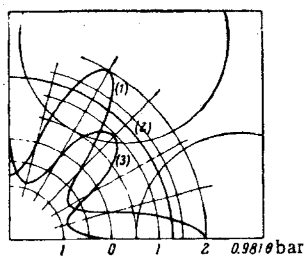

Fig. 3

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